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# Equations of Motion and Field Equations in the Five-Dimensional Unitary Theory of Relativity

*R. S. Ingarden*

In the general theory of relativity there are two different methods of deriving equations of motion from the field equations: on the one hand, the method of Einstein and Infeld[1,2] and, on the other, the method of Fock.[3,4] The difference between these two methods lies in certain differences in the views of these authors on the very essence of the general theory of relativity. Einstein and Infeld consider that "all attempts to represent matter by the energy-momentum tensor are unsatisfactory."[2] Therefore, they are concerned exclusively with field equations in empty space while representing matter as singularities of the field which must have some relation to the elementary particles of microphysics. Fock, on the contrary, considers that the general theory of relativity is a theory of gravitation alone which applies only to phenomena on an astronomical scale, and has no connection with microphysics, in which the gravitational field does not play an essential part. For this reason, Fock formulated a theory of "finite masses" and introduced the energy-momentum tensor into the field equations.

In this paper we shall endeavor to show that these two views can be reconciled to a certain extent in the five-dimensional "unitary" theory of relativity. A new view on the whole problem is thus introduced.

Let us consider the five-dimensional Riemannian geometry, with the metric

$$d\tau^2 = - g_{\mu\nu}(x^\rho)\, dx^\mu dx^\nu$$

and the signature (+, −, −, −, −). This signature is possible only if the metric form is indefinite and the variational principle

$$\delta \int d\tau = 0$$

is regular.[5] The Greek indices take the values 0, 1, 2, 3, 4, and the Latin indices the values 0, 1, 2, 3.

In order that the coordinates $x^\mu$ and the tensor components $g_{\mu\nu}$ may have [direct] physical significance, we introduce an "inertial" system of coordinates, in the sense of Fock[3,4] i.e., one which:

(1) The coordinates [sic] are harmonic (independent particular solutions of the five-dimensional D'Alembertian equation). This condition is equivalent to the coordinate conditions

$$\mathfrak{g}^{\mu\nu}_{,\mu} = 0, \tag{1}$$

where

$$\mathfrak{g}^{\mu\nu} = \sqrt{-g}\; g^{\mu\nu},\;\; g = \det g_{\mu\nu}$$

and the index after the comma indicates ordinary differentiation.

(2) The coordinate system is "the appropriate space-time coordinate system" in the sense of Hilbert[6] which corresponds to the covariant separation of the "time" $x_0$ [sic] from the "space" $(x^1, x^2, x^3, x^4)$. (We have added condition 2.)

(3) At spatial infinity $(r = [(x^1)^2 + (x^2)^2 + (x^3)^2 + c^2(x^4)^2]^{1/2} \to \infty$, where $c$ is the speed of light *in vacuo*) for every $x^0$:

$$(a)(d\tau^2)_\infty = c^2(dx^0)^2 - (dx^1)^2 - (dx^2)^2 - (dx^3)^2 - c^2(dx^4)^2 \equiv -\varepsilon_{\mu\nu}dx^\mu dx^\nu. \quad (2)$$

Also the functions $h_{\mu\nu} = -\epsilon_{\mu\nu}g_{\mu\nu}$ [sic] must tend to zero when $r \to \infty$ in such a way that the products $rh_{\mu\nu}$ remain finite;

(b) Sommerfeld's radiation condition is valid:

$$\lim_{r\to\infty}\left(\frac{\partial}{\partial r} + \frac{1}{c}\frac{\partial}{\partial x^0}\right)(rh_{\mu\nu}) = 0.$$

The conditions 1 to 3 determine the coordinate system to within a five-dimensional Lorentz transformation with constant coefficients which preserves the form (2) at infinity.[4] Thus the problem of the "general" theory of relativity has been reduced by Fock in a certain sense to the problem of the "special" theory of relativity,[4,8] but in a different sense from the attempt by Rosen,[9] which seems to us incorrect. Fock's discovery has great fundamental significance and is especially important for the definition of the concept of angular momentum[10] and for the problem of quantization of fields[11] in the general theory of relativity.

To the coordinate $x^4$ we give the physical interpretation of "proper time," and to the coordinates $x^1$, $x^2$, $x^3$ the interpretation of "proper space coordinates." These concepts, as will appear below, can be identified exactly with the corresponding concepts of the (special) theory of relativity in four dimensions (with the sole difference that in the four-dimensional theory the adjective "proper" is not used for space coordinates). We interpret $h_{\mu\nu}$ as the potentials of the "unitary field." These potentials can be grouped, according to the coordinates, into potentials of the gravitational field $h_{ij}$ and potentials of the "meso-electromagnetic" field $h_{4i} = h_i$, $h_{44} = 2h_4$.

Apart from the special interpretations of the coordinates and of the metric tensor, we base the theory on the variational principle, invariant under general transformations of the coordinates,[7]

$$\delta(S_u + S_m) = 0, \quad (3)$$

where

$$S_u = \frac{c^3}{16\pi k}\int G\sqrt{-g}\,d^5x$$

is the "5-action" of the unitary field,

$$S_m = \frac{1}{c^3}\int L\sqrt{-g}\,d^5x$$

is the "5-action" of the sources of the unitary field ("matter"), $d^5x = dx^0dx^1dx^2dx^3dx^4$, $k$ is the gravitational constant, $G = G(g_{\mu\nu}, g_{\mu\nu,\rho})$ is the Lagrangian density of the field, $L = L(g_{\mu\nu}, g_{\mu\nu,\rho}; q_A, q_{A,\mu})$ is the Lagrangian density of matter, $q_A = q_A(x^\mu)$ $(A = 1, 2, \ldots, s)$ are the "dynamical quantities" which describe the matter.

We shall show that it is possible to define the Lagrangian density of matter in the five-dimensional theory of relativity in such a way that the equations

of motion follow from the field equations without any additional assumptions concerning matter, such as equations of state. In other words, in the last analysis the only remaining dynamical quantities in the theory should be the "geometric" entities $g_{\mu\nu}$, and the variation in (3) should be performed with respect to these entities only. Thus we want the statement that the equations of motion follow from the field equations to be a postulate, and not a consequence of the theory (this postulate could be called the *geometrization postulate*). The equations of motion would not follow from the field equations if the Lagrangian density of matter depended on a greater number of dynamical variables $q_A$ than there are arbitrary functions in the transformations of coordinates ($s > 5$). For it would have been possible to obtain the equations of motion merely by varying $q_A$ in (3) ($s$ equations for $s$ functions of $q_A$). In this case, the equations of motion would have had no connection with the field equations resulting from the variation of $g_{\mu\nu}$ in (3). On the other hand, the well-known theorem of Noether[12,6] states that there are as many differential identities among the Euler-Lagrange equations of invariant variation as there are arbitrary functions in the transformation of coordinates (in our case five). Since $L$ depends, generally speaking, on $g_{\mu\nu}$ and $g_{\mu\nu,\,\rho}$ (this follows from the invariance of $S_*$), we can determine the energy-momentum tensor of matter by the usual formula:[7]

$$T_{\mu\nu} = -\frac{\delta L}{\delta g^{\mu\nu}} \equiv \frac{-2}{\sqrt{-g}} \left\{ \frac{\partial \sqrt{-g}\, L}{\partial g^{\mu\nu}} - \left( \frac{\partial \sqrt{-g}\, L}{\partial g^{\mu\nu}_{,\rho}} \right)_{,\rho} \right\}.$$

Constructing $G$, as usual, from the Ricci[7] tensor $R_{\mu\nu}$, we obtain Einstein's field equations:

$$G_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = \frac{8\pi k}{c^4} T_{\mu\nu} \tag{4}$$

($R = g^{\mu\nu} R_{\mu\nu}$), from which follow, as Noether's identities, the five equations

$$T^{\mu}_{\nu;\,\mu} = 0 \tag{5}$$

(the index after the semicolon denotes covariant differentiation). If $s = 5$, then Eq. (5) (together with the boundary conditions and Eqs. (1) and (4), determining $g_{\mu\nu}$) will be sufficient, generally speaking, to determine $q_A$. Therefore, (5) will then be the equations of motion of matter resulting from the field-equations (4).

In the general theory of relativity, one usually employs an energy-momentum tensor of matter of the form[7]

$$T_{\mu\nu} = (p + \varepsilon)\, u_\mu u_\nu + p g_{\mu\nu} \tag{6}$$

($p$ is the pressure, $\epsilon$ the density, $u_\mu$ the speed of matter) and

$$u_\mu u^\mu = -1. \tag{7}$$

Tensor (6) contains one extra function ($s = 6$, taking (7) into account). This is the source of difficulties which cannot be overcome in four dimensions since in the general case it is impossible to eliminate any of these functions. This obstacle forced Einstein and Infeld to abandon the energy-momentum tensor and led Fock to limit the application of this tensor to microphysics, where one can assume the existence of a thermodynamical equation of state, which relates $p$ to $\epsilon$ and is different in different cases. In the five dimensional world, however, it is possible, without contradicting experiment, to assume

the following universal relation, which seems to us the simplest and most natural:

$$T^{\mu}_{\mu} = 0. \tag{8}$$

From the above and from (6), we obtain the equation $\epsilon = 4p$, which in the four-dimensional theory corresponds to the "equation of state"[7] of radiation: $\epsilon = 3p$. The energy-momentum tensor takes the form

$$T_{\mu\nu} = p\,(5u_{\mu}u_{\nu} + g_{\mu\nu}). \tag{9}$$

In (9) we have six functions of the type $q_A$, namely $p(x^{\rho})$ and $u_{\mu}(x^{\rho})$, five of which, in view of (7), are independent ($s = 5$, as it should).

From (8) and (4) there follows the vanishing of the scalar curvature, $R = 0$, i.e., the same property which exists in Einstein and Infeld's theory (where $T_{\mu\nu} = 0$). Our field equations can be written

$$R_{\mu\nu} = \frac{8\pi k}{c^4}\,p\,(g_{\mu\nu} + 5u_{\mu}u_{\nu}). \tag{10}$$

Field equations of the type (10) seem to be a natural generalization of the field equations $R_{\mu\nu} = 0$. This generalization is different from the Einstein-deSitter generalization, $R_{\mu\nu} = \lambda g_{\mu\nu}$ ($\lambda$ being the "cosmological constant"), which was recently exhaustively studied by Petrov.[13]

On the other hand, it follows from (9) that the "field of matter" $p$, $u_{\mu}$ does not have a "rest mass" in the five-dimensional sense, and propagates along the five-dimensional "light-cone" $d\tau^2 = 0$. Therefore, the "particles" of this field (which are points of great density or else singularities of the field in the classical sense, or are particles in the quantum sense) must be "five-dimensional photons." In particular, in a flat space, we have

$$c^2\,(dx^4)^2 = c^2\,(dx^0)^2 - (dx^1)^2 - (dx^2)^2 - (dx^3)^2,$$

which justifies our interpretation of $x^4$ as "proper time" in the sense of the four-dimensional theory of relativity. Our interpretation of the fifth coordinate of space differs from the one given by Fock[14] and Rumer[15] but leads to the familiar interpretation of the motion of particles as the motion of "photons" in a five-dimensional world.[16-18] Yet, as far as we know, all the papers written to date are based on the premises of Kaluza's[19] and Klein's[20] theories, which are somewhat more restricted than ours.

We shall prove elsewhere that it is possible to build a quantum theory of elementary particles on the classical basis here indicated.

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